Ribbon Concordance Quandles

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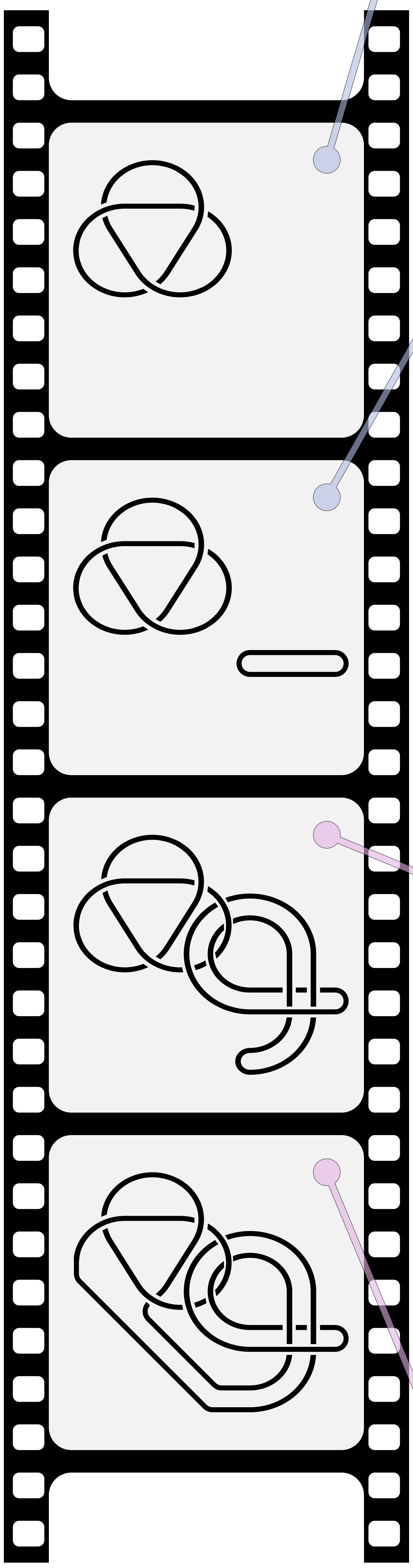


We begin with ribbon concordances

Prologue: knots and links

A *(classical) link* L with n components is the image of an embedding $\bigsqcup_{i=1}^{n} \mathbb{S}^1 \hookrightarrow \mathbb{S}^3$. A link with a single component is called a *knot*. A *diagram* of a link is the image of a generic projection onto a plane along with markings of under and overcrossings.

Example 1.

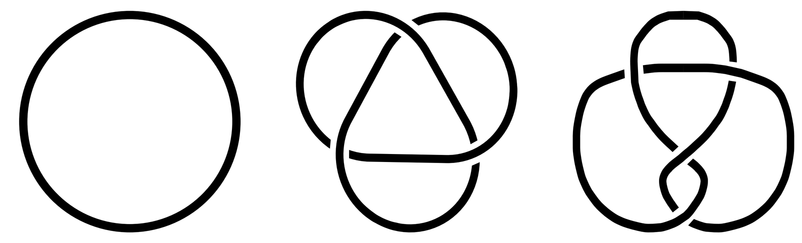


Figure 1: Examples of knot diagrams.

Example 2.

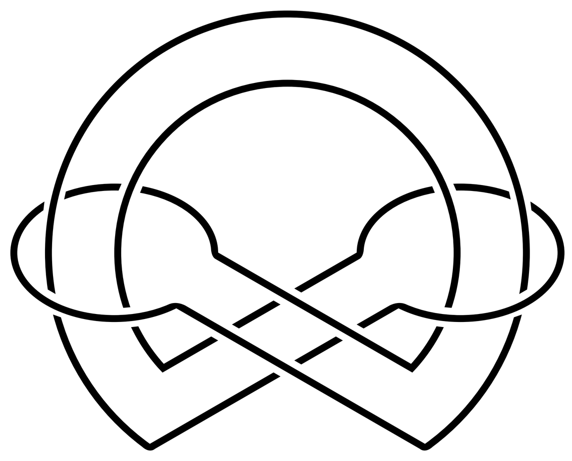


Figure 2: Stevedore's knot, a well-known ribbon knot.

Antagonist: ribbon concordance [3]

A *concordance* C between two knots K_0 and K_1 is an embedded annulus

$$(\mathbb{S}^1 \times I, \mathbb{S}^1 \times \{0\}, \mathbb{S}^1 \times \{1\}) \hookrightarrow (\mathbb{S}^3 \times I, K_0, K_1).$$

If the restriction of the projection $\mathbb{S}^3 \times I \to I$ to C (assumed to be a Morse function) has no critical points of index 2 (local maxima), then C is called a *ribbon concordance* from K_1 to K_0. A knot is *slice* if it is concordant to the unknot and *ribbon* if there exists a ribbon concordance from it to the unknot.

Result conjectured by Gordon [3] in the 80s, proven by Agol in 2022 [1]: ribbon concordance of knots is a partial order.

We introduce a new component: quandles

Protagonist: quandle

A *quandle* Q is a set equipped with two binary operations $\triangleright$ and $\triangleleft$ satisfying the following three axioms:

(1) $x \triangleright x = x = x \triangleleft x$ for all $x \in Q$,

(2) $(x \triangleright y) \triangleleft y = x = (x \triangleleft y) \triangleright y$ for all $x, y \in Q$,

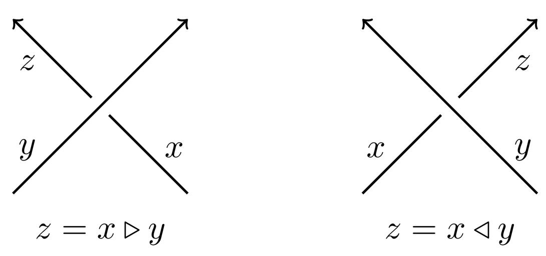
(3) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ and $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ for all $x, y, z \in Q$.

Much like with groups, we define a *quandle presentation* $\langle X \mid R \rangle$ with generators X and relations R.

Example 3. Some algebraic examples of quandles:

- On a group G, define $g \triangleright h = h^{-1}gh$ and $g \triangleleft h = hgh^{-1}$. We obtain the *conjugation quandle* $\mathrm{Conj}(G)$.
- On a module M over $\mathbb{Z}[t, t^{-1}]$, define $a \triangleright b = ta + (1-t)b$ and $a \triangleleft b = t^{-1}a + (1-t^{-1})b$. We obtain the *Alexander quandle* $A_t(M)$.
- On $\mathbb{Z}_n$, define $x \triangleright y = 2y - x = x \triangleleft y$. We obtain the *dihedral quandle* R_n.

Example 4. Given a diagram of a knot K, we compute the fundamental quandle $Q(K)$ of K by taking the arcs as generators and taking relations from the crossings:



Theorem [4]: for any two knots K and K', if $Q(K) \cong Q(K')$, then K' is ambient isotopic to K or to the reversely oriented mirror image of K.

Flashback: The fundamental quandle of a codimension two embedded manifold [2]

Let $L \subset M$ be a codimension 2 manifold embedding, let N_L be the normal disk bundle around L and let $E_L = \mathrm{cl}(M - N_L)$ be its exterior. Define

$$\Gamma_L := \frac{\mathcal{C}((I, \{0\}, \{1\}) \to (E_L, \partial N_L, \{z\}))}{\text{homotopy}},$$

where $z \in E_L$ is a choice of a basepoint. The *fundamental quandle* $Q(L)$ of L is the set Γ_L together with operations

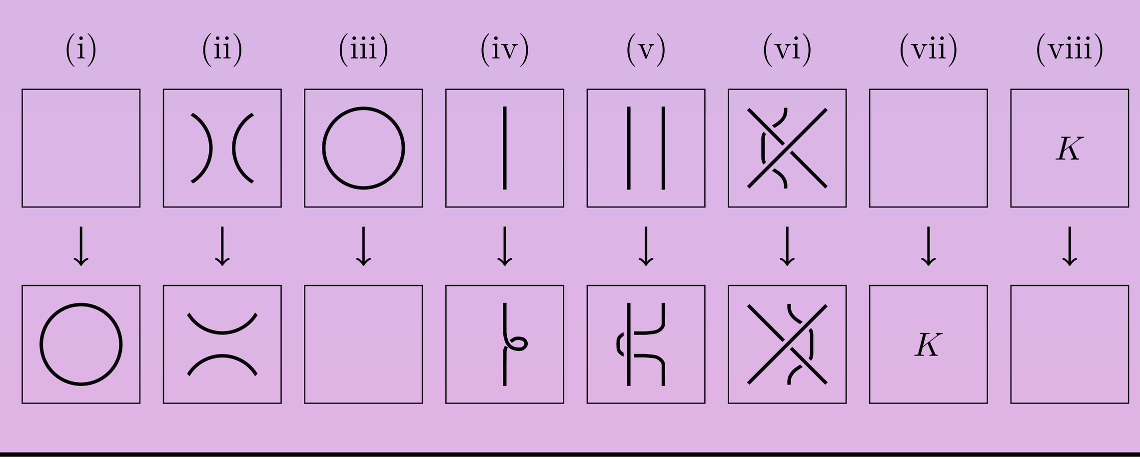
$$[\alpha] \triangleright [\beta] := [\alpha \cdot \overline{\beta} \cdot m_{\beta(0)} \cdot \beta] \text{ and } [\alpha] \triangleleft [\beta] := [\alpha \cdot \overline{\beta} \cdot \overline{m_{\beta(0)}} \cdot \beta],$$

where $\cdot$ denotes concatenation of paths and m_p is the meridian based at p. This is an ambient isotopy invariant.

We mix up concordances and quandles

Exposition: motion pictures

Any compact surface F in $\mathbb{R}^4$ (or $\mathbb{S}^3 \times I$) can be described by a *motion picture* consisting of a finite number of classical links called *frames*. These are the fibres of regular values of the projection onto the last coordinate, assumed to be a Morse function that is constant on every boundary component of F. The diagrams of two subsequent frames are related by planar ambient isotopy and one of the following local moves: (i)-(iii) are 0-handle, 1-handle and 2-handle addition, respectively, (iv)-(vi) are Reidemeister moves, (vii) is the beginning of a new component with boundary K and (viii) is the ending of a component with boundary K.



Escalation: computing the fundamental quandle from a motion picture

To compute the fundamental quandle of a surface F, start with the fundamental quandle of the first frame and modify it according to the moves used to obtain subsequent frames:

(i) add a generator x,

(ii) add a relation between the two elements represented by arcs that the 1-handle connects,

(vii) add the generators and relations corresponding to the new component of the boundary,

other: do nothing.

Example 5. The fundamental quandle of the ribbon concordance C represented by the big motion picture is

$$Q(C) = \langle a, b, x \mid a \triangleright b = (x \triangleleft a) \triangleleft x \rangle.$$

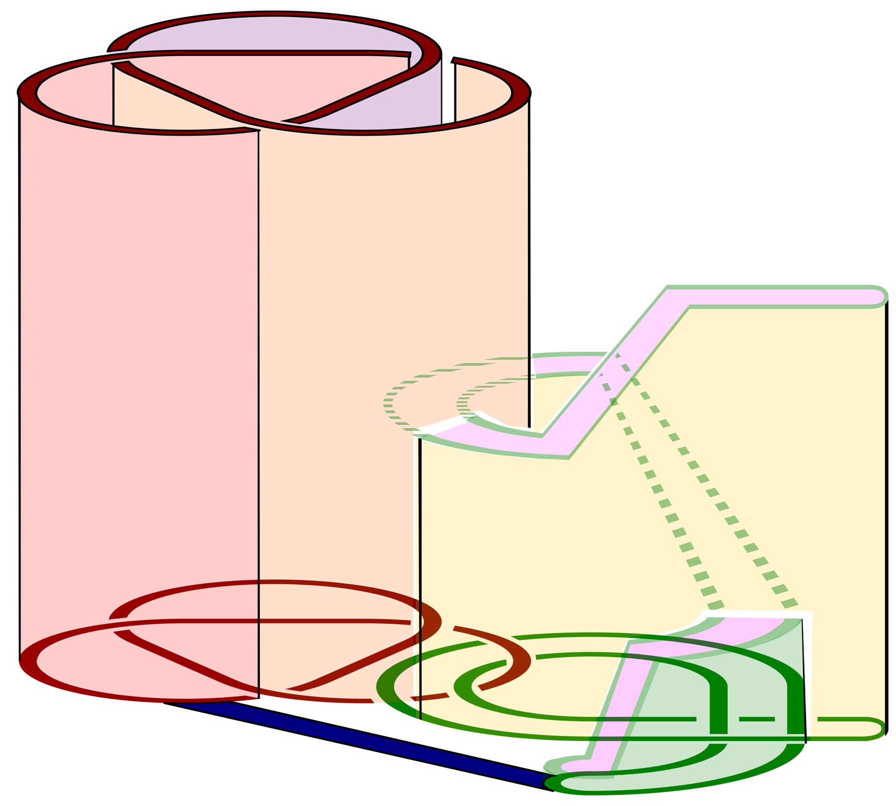


Figure 3: A diagram of the ribbon concordance represented by the big motion picture.

We bridge the notions

Climax: main theorem

Let C be a ribbon concordance from K_1 to K_0. Then there exists a monomorphism $Q(K_0) \to Q(C)$ and an epimorphism $Q(K_1) \to Q(C)$.

Resolution: obstructions to ribbon concordances

Using the theorem on knots with simple fundamental quandles (such as torus knots), we can obtain obstructions to ribbon concordances between them (in theory, we can use the theorem for any pair of knots, but working with quandles is hard).

Sequel: future work

Can this result be used to obtain obstructions to ribbon concordances using more computable quandle-related invariants, such as cocycle invariants and colouring invariants?

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[2] Roger Fenn and Colin Rourke. 'Racks and links in codimension 2'. In: *J. Knot Theory Ramifications* 01 (1992). DOI: 10.1142/S0218216592000203.

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[4] David Joyce. 'A classifying invariant of knots, the knot quandle'. In: *J. Pure Appl. Algebra* 23.1 (1982), pp. 37–65. DOI: 10.1016/0022-4049(82)90077-9.